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The Aeroelastic Effects of Transverse Shear Deformation on Composite Wings in Various Speed Flow Regimes

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Abstract

This project analyzes the effect of transverse shear deformation upon the aeroelastic response of composite wings in high speed flow regimes. Previously, models have been developed to predict the aeroelastic characteristics of classical materials in high speed flow. However, these studies ignored transverse shear by assuming an infinite modulus of rigidity. This assumption underestimates transverse flexibility by ignoring the transfer of loads through the wing thickness. By assuming a finite modulus of rigidity and redeveloping the governing equations, this model would more accurately predict the aeroelastic response of composite wings. This present analysis concerns mainly the determination of aeroelastic trends vice more detailed solutions. Thus, linearized flow theory is used. Upon conclusion, this study gives results for divergence speed and flutter speeds, as well as their mode shapes.

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List of Symbols

AR .... aspect ratio
a .... lift curve slope
b .... wingspan
c .... chord length
c_{mac} .... pitching moment about aerodynamic center
d .... distance between center of mass and elastic axis
E .... Young's modulus
e .... distance between aerodynamic center to elastic axis
G' .... transverse shear modulus
h .... plunging displacement
$I^{(m,n)}$ .... generalized mass
L .... sectional lift
l .... wing semi-span
Q_{ij} .... elastic moduli
Q_n .... normalized dynamic pressure
q_n .... dynamic pressure normal to quarter-chord line
T .... sectional torque
$T_{ij}^{(m,n)}$ .... generalized stress couples
t .... time
u_1 .... chordwise displacement
u_2 .... spanwise displacement
u_3 .... transverse displacement
x_0 .... position of elastic axis
x_1 .... chordwise coordinate
x_2 .... spanwise coordinate
x_3 .... transverse coordinate

α_0 .... rigid angle of attack
γ_{ij} .... shearing strain component
δ .... variation sign
ϵ_{ij} .... normal strain component
η .... dimensionless spanwise coordinate
θ .... twist about the pitch axis
Λ .... wing sweep angle
ν .... Poisson's ratio
ψ_1 .... angle of rotation about x_2
ψ_2 .... angle of rotation about x_1
ω_b .... natural bending frequency
σ_{ij} .... component of stress tensor
$\tau_i^{(m,n)}$ .... generalized body force

Chapter 1

Background

Since the days of the Wright Flyer, the aviation industry has constantly searched for lighter, stronger, and more durable materials. In the past century this pursuit has taken aeronautical engineers from cloth and wood to paper thin steel to honeycombed aluminum structures. The next logical step in this evolution is composite materials. However, due to the differences in the properties of composites and the so-called classical materials, designers need guidance to predict the characteristics of composite aeroelastic structures.

The purpose of this project is to develop a model, which will predict the impact of using composite materials in aircraft wings. In particular, it will look at the aeroelastic effects caused by transverse shear deformation.

1.1 A Brief Overview of Aeroelasticity

Often, the complexities of design force engineers to break down their work into simpler components. In aeronautical engineering, the two most vital of these components are the aerodynamic and structural characteristics of the aircraft. These basic pieces of the puzzle answer the questions: a) will the aircraft fly? and b) will it stay in one piece? In a perfect undergraduate level world, answering these two questions would be enough. However, in

the real world things are not as simple. In truth the aerodynamic and structural forces on an aircraft are dependent upon each other. This being said, the next logical question becomes, "How do the two affect each other?" In response to this, engineers began the study of aeroelasticity.

Aeroelasticity can be defined as the study of the interaction between the aerodynamic, inertial, and structural forces acting on an object.[1,3] Taking principles from the fields of aerospace and mechanical engineering, the aeroelastician is primarily concerned with determining the effects of placing an aerodynamic load on a structure. Although usually associated with aircraft design, aeroelasticians are used in a variety of fields. For instance, the most famous of all aeroelastic failures occurred on the Tacoma Narrows Bridge, which can be seen in Fig. 1.1. In 1940, the aerodynamic load caused by wind blowing through the

Figure 1.1: Torsional oscillation of Tacoma Narrows Bridge.[2]

valley caused this suspension bridge, constructed of concrete and steel, to twist, bend, and eventually collapse, as if it were made of rubber. The failure of the Tacoma Narrows Bridge can be attributed to what is known as flutter instability.[(3,1.7)] Although not all aeroelastic events are as visually dramatic, they do occur, and ignoring them can potentially lead to catastrophic results.

The study of aeroelasticity can be broken down into two major branches: static and dynamic. The case of static aeroelasticity is concerned with systems in equilibrium. Once an aerodynamic load is placed on an aircraft, its structures will deform, redistributing the load. This can lead to one of two possibilities. The first is simply a new state of equilibrium in which the aerodynamic and structural characteristics of the wing are slightly changed for better or worse. The second, and less appealing, case is that the redistributed loads will escalate until the wing fails. This is known as static divergence.

Dynamic aeroelasticity is concerned with time dependent instabilities. These can be either transient or oscillatory depending upon the nature of the response. The most prominent of these cases is flutter, which simply represents a harmonically oscillating wing and constitutes the stability boundary between damped and undamped oscillations. Aeroelastic methods can be used to predict whether the system will eventually stabilize or diverge, as in the case of the Tacoma Narrows Bridge.[3]

1.2 Composites vs. Classical Materials

In recent years, a continual improvement in composite materials has given engineers a new level of freedom in design. These new materials allow for lightweight, high performance structures, which could not have been constructed from metals. Metals are known as the "classical" materials. However, with this change from metallic to composite structures, the classic structural models must be reexamined to determine if they can be accurately applied

to these new-age materials.

The ability to tailor materials to increase their functionality truly lends itself to the field of aircraft and spacecraft design. In the quest to go faster and fly higher, weight and structural stability are vital components. Over the past two decades, the effects of these materials on the aeroelastic behavior of aircraft wings have been examined. Although composite technology is allowing for the creation of wing structures of enhanced efficiency, the incorporation of the new technology is forcing aeroelasticians to look back upon the old models.

In many cases, the classical models developed over the past sixty years need only to be extended, as the composite materials' behavior closely resembles that of the metals. However, composites tend to differ greatly from metals in the case of transverse shear flexibility. Shear effects are created from the transfer of loads through the thickness of a structure. The greater the shear rigidity, the less a material will transversely deform under a given load. Early on in the derivation of the classical model, the assumption of infinite rigidity in transverse shear was made. This assumption, referred to as Kirchhoff's hypothesis, can be made and justified in metallic structures. Conversely, composite materials have been shown to have a much lower modulus of transverse shear rigidity.[4, 10] In some cases the effects of transverse shear deformation have been shown to affect the static aeroelastic response of a composite wing by almost fifty percent.[5,790] This is obviously not negligible. Due to the large impact of transverse shear upon composite materials, the classical model cannot be used or even extended to accurately predict the aeroelastic characteristics of composite wings, but rather a new model must be created taking into account a finite rigidity in transverse shear deformation.

Chapter 2

Procedure

In order to accurately develop a mathematical model representative of an aircraft wing in high-speed flow, a four-step process is required. First, the equations of motion must be developed for the system. Second, the structural mechanics of the wing must be inserted into these equations. Next, the aerodynamic loads must be included to create a system of governing equations. Finally, this system must be solved to determine the critical aeroelastic eigenvalues and mode shapes of the wing. In order to accomplish this last step, the differential equation solver in *MATHEMATICA* was used.

2.1 Equations of Motion

The first step in analyzing any physical system is to choose a set of axes, which can accurately represent that system. Fig. 2.1 illustrates the geometric model of a generic swept wing. The wing geometry is described by a Cartesian coordinate system with x_1 set as the effective wing root, x_2 set normal to x_1 as the reference axis along the wingspan, and x_3 describing the normal direction to the wing surface. This three dimensional system will be used to describe the wing displacement under static and dynamic loads resulting from airflow over the wing. Λ is used to describe the angle of sweep with positive angles associated with swept

Figure 2.1: Geometry of a generic wing.[5,788]

back wings and negative angles describing a forward sweep. This becomes significant as Ref. [5] shows that even in low speed flow, the angle of sweep has a dramatic effect on transverse shear. Due to the fact that span is much greater than chord and thickness in most common wings, the three dimensional wing will be reduced to a one-dimensional system with only a spanwise variation of properties.

Having defined the coordinate system to be used, the equations of motion can now be developed. Assuming first-order transverse shear deformation, that is the displacement field varying linearly through the thickness, the three-dimensional time dependent displacement equations are given by:

$$U_1(x_1, x_2, x_3; t) = u_1(x_1, x_2; t) + x_3\psi_1(x_1, x_2; t) \tag{2.1}$$

$$U_2(x_1, x_2, x_3; t) = u_2(x_1, x_2; t) + x_3\psi_2(x_1, x_2; t) \tag{2.2}$$

$$U_3(x_1, x_2, x_3; t) = u_3(x_1, x_2; t) \tag{2.3}$$

In Equations (2.1-2), the first terms on the right-hand side represent the displacement components in the reference plane ($x_3 = 0$), while the second terms represent the displacement off the reference plane. ψ_1 is twist about the x_2 axis, and ψ_2 is twist about the x_1 axis. Equation (2.3) shows that the normal displacement of any point in the wing structure is assumed to be the same as a point in the reference plane. Thus, there is a constant wing thickness during deformation.

To further simplify the system, it is assumed that chordwise rigidity exists. Thus, $u_1 \to 0$, $u_2 \to u_2(x_2; t)$, and $\psi_1 \to \psi_1(x_2; t)$, which is defined as $\theta(x_2; t)$, twist about the pitching axis.

The next step is to rewrite this three-dimensional displacement field as a one-dimensional system. This is a valid assumption as long as the wing's span is much larger than its chord, as it resembles a beamlike structure; thus, the larger the aspect ratio, the more accurate the model. The rotation about the x_1 axis, $\psi_2(x_1, x_2; t)$, is modeled as a linear function in x_1. $\bar{\psi}_2$ is the rotation of the reference axis about x_1, while $\tilde{\psi}_2$ is the rotation of the wing elements off the reference axis. This can be written as

$$\psi_2(x_1, x_2; t) = \bar{\psi}_2(x_2; t) + x_1\tilde{\psi}_2(x_2; t) \tag{2.4}$$

The normal displacement of the reference plane can be written as

$$u_3(x_1, x_2; t) = h(x_2; t) - (x_1 - x_0)\theta(x_2; t) \tag{2.5}$$

where $h(x_2; t)$ is the vertical displacement due to plunging as measured at the elastic axis.

Inserting Equations (2.4-5) into Equations (2.1-3) results in the one-dimensional displacement components. These equations are now only a function of time and the spanwise coordinate, and can be written as

$$U_1 = x_3\theta \tag{2.6}$$

$$U_2 = u_2 + x_3(\bar{\psi}_2 + x_1\tilde{\psi}_2) \tag{2.7}$$

$$U_3 = h - (x_1 - x_0)\theta \tag{2.8}$$

2.2 Hamilton's Principle

Although seemingly complex, the entire method used to find the wing modes is based on the principle of conservation of energy. Basically, the kinetic energy generated by the unsteady aerodynamic loads is transfered to the wing creating potential energy in the form of a strain. The structural analysis then determines how the wing will react under these loads, either deforming into a new state of equilibrium or failing. The equations of motion result from the application of Hamilton's variational principle. Minimizing the function

$$\delta J = 0$$

$$\delta J = \int_{t_0}^{t_1} \Big\{ -\int_{\varphi} \sigma_{ij}\delta U_{ij} d\varphi + \int_{\varphi} \rho(H_i - \ddot{U}_i)\delta U_i d\varphi + \int_{\Omega_\sigma} \underline{\sigma_i}\delta U_i d\Omega \Big\}\, dt \tag{2.9}$$

The first term on the right-hand side represents the strain energy present inside the structure of the wing. σ_{ij} corresponds to the internal stresses, and φ represents the volume of the wing. The second term represents the body forces and kinetic energy of the system, with ρ being the density of the structure. The final term denotes the energy resulting from the surface stresses, $\underline{\sigma_i}$. Ω_σ is the wing's surface area.

Hamilton's Principle is an application of variational calculus and states that a physical system will go from one state to the next through the lowest possible change in energy. By integrating over time and setting the change in energy of the system equal to zero, the state of deformation at t_1 can be determined. Because the variational, δU does not necessarily have to equal zero, its coefficients do. Thus, from inserting the structural properties and aerodynamic forces into Equation (2.9), the displacements at a given state can be found.

2.3 Structural Analysis

It is now necessary to insert the structural properties of the wing. However, due to the fact that the displacements are still general components at this point, Equation (2.9) must be rewritten in terms of strain in order to simplify the system. The following relations are used to accomplish this.(7,147)

$$\epsilon_{11} = U_{1,1} = 0 \tag{2.10}$$

$$\epsilon_{22} = U_{2,2} = u_2' + x_3(\bar{\psi}_2' + x_1\tilde{\psi}_2') \tag{2.11}$$

$$\epsilon_{33} = U_{3,3} = 0 \tag{2.12}$$

$$\gamma_{12} = U_{1,2} + U_{2,1} = x_3(\theta' + \tilde{\psi}_2) \tag{2.13}$$

$$\gamma_{13} = U_{1,3} + U_{3,1} = 0 \tag{2.14}$$

$$\gamma_{23} = U_{2,3} + U_{3,2} = \bar{\psi}_2 + x_1\tilde{\psi}_2 + h' + (x_0\theta)' - x_1\theta' \tag{2.15}$$

Here the subscripts (i, j) indicate differentiation of the i^{th} component with respect to the j^{th} variable. Of particular interest to this model is Equation (2.15). This is where transverse shear deformation can be accounted for. Had $\gamma_{23} = 0$, Kirchoff's Hypothesis would be satisfied as γ_{13} also equals zero. Recall that Kirchoff's hypothesis is the assumption made

for classical materials in which transverse shear rigidity is infinite, causing transverse shear strains to vanish.

After the strain relations are inserted, Equation (2.9) becomes seemingly more complex. In order to more easily deal with it, the following relations are defined

$$T_{ij}^{(m,n)}(x_2) = \int_A \sigma_{ij} x_1^m x_3^n \; dA \tag{2.16}$$

$$\tau_{ij}^{(m,n)}(x_2) = \int_A \rho H_i x_1^m x_3^n \; dA \tag{2.17}$$

$$I^{(m,n)}(x_2) = \int_A \rho x_1^m x_3^n \; dA \tag{2.18}$$

Equations (2.16-8) represent the generalized stress couples, body forces, and mass respectively. From these, the most general equations of motion can be derived. These are[5]

$$\delta u_2 : \qquad I^{(0,0)}\ddot{u}_2 + I^{(0,1)}\ddot{\bar{\psi}}_2 + I^{(1,1)}\ddot{\tilde{\psi}}_2 - T_{22}^{(0,0)\prime} - \tau_2^{(0,0)} = 0 \tag{2.19}$$

$$\delta\bar{\psi}_2 : \qquad I^{(0,1)}\ddot{u}_2 + I^{(0,2)}\ddot{\bar{\psi}}_2 + I^{(1,2)}\ddot{\tilde{\psi}}_2 - T_{22}^{(0,1)\prime} + T_{23}^{(0,0)} - \tau_2^{(0,1)} = 0 \tag{2.20}$$

$$\delta\tilde{\psi}_2 : \; I^{(1,1)}\ddot{u}_2 + I^{(1,2)}\ddot{\bar{\psi}}_2 + I^{(2,2)}\ddot{\tilde{\psi}}_2 - T_{22}^{(1,1)\prime} + T_{12}^{(0,1)} + T_{23}^{(1,0)} - \tau_2^{(1,1)} = 0 \tag{2.21}$$

$$\delta h : \qquad I^{(0,0)}\ddot{h} - (I^{(1,0)} - x_0 I^{(0,0)})\ddot{\theta} - T_{23}^{(0,0)\prime} - L - \tau_3^{(0,0)} = 0 \tag{2.22}$$

$$\begin{aligned} \delta\theta : \qquad & (I^{(0,2)} + I^{(2,0)} - 2x_0 I^{(1,0)} + x_0^2 I^{(0,0)})\ddot{\theta} - (I^{(1,0)} - x_0 I^{(0,0)})\ddot{h} - T_{12}^{(0,1)\prime} \\ & + T_{23}^{(1,0)\prime} - x_0 T_{23}^{(0,0)\prime} - T - \tau_1^{(0,1)} + \tau_3^{(1,0)} - x_0 \tau_3^{(0,0)} = 0 \end{aligned} \tag{2.23}$$

The natural boundary conditions also result from Hamilton's variational principle, and are given by

Root conditions ($x_2 = 0$):

$$u_2 = \bar{\psi}_2 = \tilde{\psi}_2 = h = \theta = 0 \tag{2.24}$$

Tip conditions ($x_2 = l$):

$$T_{22}^{(0,0)} = \bar{T}_{22}^{(0,0)} \tag{2.25}$$

$$T_{22}^{(0,1)} = \bar{T}_{22}^{(0,1)} \tag{2.26}$$

$$T_{22}^{(1,1)} = \bar{T}_{22}^{(1,1)} \tag{2.27}$$

$$T_{23}^{(0,0)} = \bar{T}_{23}^{(0,0)} \tag{2.28}$$

$$T_{12}^{(0,1)} - T_{23}^{(1,0)} = \bar{T}_{12}^{(0,1)} - \bar{T}_{23}^{(1,0)} \tag{2.29}$$

The system, Equations (2.19-29), represents the most general one-dimensional system. The only assumptions made up to this point have been chordwise rigidity and constant thickness. Now, the assumption that the wing is composed of a single layer of composite is made. This assumption eases the process of determining the stress resultants, and if deemed necessary, future study can be undertaken on wings with more than one layer.

The next step is to begin to define the material properties of the system in order to analyze a more specific wing. By now inserting the constitutive equations, the generalized stress couples can be put in terms of the displacement components. The three-dimensional constitutive equations are: [4,51]

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{21} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{61} & \bar{Q}_{62} & \bar{Q}_{66} \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix} \tag{2.30}$$

$$\begin{pmatrix} \sigma_{13} \\ \sigma_{23} \end{pmatrix} = \begin{pmatrix} \bar{c}_{45} & \bar{c}_{45} \\ \bar{c}_{54} & \bar{c}_{55} \end{pmatrix} \begin{pmatrix} \gamma_{13} \\ \gamma_{23} \end{pmatrix} \tag{2.31}$$

Introduction of the three-dimensional constitutive equations into the one-dimensional stress tensors can be done through the following relations:

$$[\bar{A}_{ij}(x_2), a_{ij}(x_2), \bar{a}_{ij}] = \int_c A_{ij}[1, x_1, x_1^2]\ dx_1 \tag{2.32}$$

$$[\bar{B}_{ij}(x_2), b_{ij}(x_2), \bar{b}_{ij}] = \int_c B_{ij}[1, x_1, x_1^2]\ dx_1 \tag{2.33}$$

$$[\bar{D}_{ij}(x_2), d_{ij}(x_2), \bar{d}_{ij}] = \int_c D_{ij}[1, x_1, x_1^2]\ dx_1 \tag{2.34}$$

where

$$[A_{ij}(x_1, x_2), B_{ij}(x_1, x_2), C_{ij}(x_1, x_2)] = \int_0^t \bar{Q}_{ij}[1, x_3, x_3^2]\ dx_3 \tag{2.35}$$

Equations (2.30-1) represent any material containing monoclinic symmetry, i.e. symmetry with respect to the vertical coordinate. From these principles comes higher level aeroelastic theory, such as structural tailoring. However, further refinement of the structural model can be used as the current emphasis is being placed on general trends, not exact solutions. With this in mind, further assumptions can be made. In order to reduce the equations to a more manageable size, it is assumed that the structural properties are constant in the spanwise direction and also that the reference axis is along the wing mid-chord. Now the generalized stress resultants can be written as:

$$T_{22}^{(0,0)} = \bar{A}_{22} u_2' \tag{2.36}$$

$$T_{22}^{(0,1)} = \bar{D}_{22} \bar{\psi}_2' + \bar{D}_{26} \theta_2' + \bar{D}_{26} \tilde{\psi}_2 \tag{2.37}$$

$$T_{23}^{(0,0)} = \bar{A}_{55} \bar{\psi}_2 + \bar{A}_{55} h_2' + \bar{A}_{55} (x_0 \theta)_2' \tag{2.38}$$

$$T_{22}^{(1,1)} = \bar{d}_{22} \tilde{\psi}_2' \tag{2.39}$$

$$T_{12}^{(0,1)} = \bar{D}_{62} \bar{\psi}_2' + \bar{D}_{66} \theta' + \bar{D}_{66} \tilde{\psi}_2 \tag{2.40}$$

$$T_{2,3}^{(1,0)} = \bar{a}_{55} \tilde{\psi}_2 - \bar{a}_{55} \theta' \tag{2.41}$$

The final step in developing the wing's structural model is to replace the one-dimensional stiffness quantities with the material properties. This can be done in terms of three properties: Young's modulus, the modulus of rigidity, and Poisson's ratio. Young's modulus, E, is

a measure of the ratio of stress-to-strain in a material and gives a comparison between the stiffnesses of various materials. The modulus of rigidity, G relates shearing stress-to-shearing strain in much the same manner. Poisson's ratio, ν, relates the axial strain to the lateral strain. As can be guessed, all three properties are related. However, since each measures a slightly different property, it is necessary to include all three in the material analysis. This being said, the one-dimensional stiffness quantities can now be written as[4,53]

$$\bar{A}_{55} = tcG \tag{2.42}$$

$$\bar{a}_{55} = \frac{G}{12}c^3 t \tag{2.43}$$

$$\bar{D}_{22} = \frac{E}{12(1-\nu^2)}ct^3 \tag{2.44}$$

$$\bar{d}_{22} = \frac{E}{144(1-\nu^2)}c^3 t^3 \tag{2.45}$$

$$\bar{D}_{26} = 0 \tag{2.46}$$

$$\bar{D}_{66} = \frac{E}{24(1+\nu)}ct^3 \tag{2.47}$$

where t is the wing thickness, and c is the chord length.

The last set of assumptions to be made concern the body forces and inertial terms. Because the wing deformation is measured due to the aerodynamic loads and not the weight of the wing, the body forces will be ignored. Doing this significantly reduces the complexity of the governing equations and boundary conditions. Also, at this point the system can be reduced to four governing equations and eight boundary conditions by substitution.

With all the structural pieces in place, the governing equations can now be written in terms of the displacement components, the structural properties, and the aerodynamic forces. The governing system of aeroelastic equations is given by[5,789]

$$\delta\bar{\psi}_2 : \quad -\frac{E}{12(1-\nu^2)}ct^3\bar{\psi}_2'' + tcG\bar{\psi}_2' + tcGh' + x_0tcG\theta' = 0 \tag{2.48}$$

$$\delta\tilde{\psi}_2 : \quad -\frac{E}{144(1-\nu^2)}c^3t^3\tilde{\psi}_2'' + \Big(\frac{E}{24(1+\nu)}ct^3 - \frac{G}{12}c^3t\Big)\theta' + \Big(\frac{E}{24(1+\nu)}ct^3 + \frac{G}{12}c^3t\Big)\tilde{\psi}_2 = 0 \tag{2.49}$$

$$\delta h : \quad tcG\tilde{\psi}_2' + tcGh' + x_0tcG\theta'' + L = 0 \tag{2.50}$$

$$\delta\theta : \quad \Big(\tfrac{E}{24(1+\nu)}ct^3 + \tfrac{G}{12}c^3t + x_0^2tcG\Big)\theta'' + \Big(\tfrac{E}{24(1+\nu)}ct^3 - \tfrac{G}{12}c^3t\Big)\tilde{\psi}_2'$$

$$+x_0tcG\bar{\psi}_2' + x_0tcGh'' + T = 0 \tag{2.51}$$

with the boundary conditions:

$x_2 = 0$:

$$\bar{\psi}_2 = \tilde{\psi}_2 = h = \theta = 0 \tag{2.52}$$

$x_2 = l$:

$$\bar{\psi}_2' = \tilde{\psi}_2' = 0 \tag{2.53}$$

$$\bar{\psi}_2 + h' + x_0\theta' = 0 \tag{2.54}$$

$$\Big(\frac{E}{24(1+\nu)}ct^3 - \frac{G}{12}c^3t\Big)\tilde{\psi}_2 + \Big(\frac{E}{24(1+\nu)}ct^3 + \frac{G}{12}c^3t\Big)\theta' = 0 \tag{2.55}$$

After a good deal of math and material science, the structural model for the wing is now complete, but this is only the first half of the development. The next step in the process is to blend the aerodynamics into this model. This blend of disciplines is what makes the study of aeroelasticity difficult, and at the same time, interesting.

2.4 Static Aerodynamic Forces

In order to complete the governing equations, the aerodynamic load and torque per unit span must be inserted into the governing equations. Due to the fact that structural instability is the amplification of small deformations in the wing, the aeroelastic response to large deformations represents a post-stability analysis, which is unnecessary. This allows for the use of linear aerodynamic theory. Although not exact, the linear equations for load and torque provide enough accuracy for this study.

In aerodynamic theory there are two very different types of flow: steady and unsteady. Steady flow can be assumed along constant flight paths through laminar fluids, and its properties are independent of time.[8] Obviously from the mathematical standpoint, this is the more desirable type of flow. From these equations, the static aeroelastic equations can be determined. Solving this system allows for calculation of the flow velocity at which the static instability, known as wing divergence, will occur, as well as the wing mode shapes obtained for sub-critical flow.

In the case of steady flow, lift per unit span can be obtained by

$$L(x_2) = q_n ac(\alpha_0 + \theta - h' tan\Lambda) \tag{2.56}$$

Equation (2.56) shows that much of the lift is determined by known properties of the system, such as dynamic pressure, lift curve slope, chord length, and angle of attack. However, notice that θ and h' come into this equation. These terms represent the aeroelastic interaction between the aerodynamics and the structural properties. The terms in parenthesis together are known as the effective angle of attack. This term accounts for the angle of

attack at the wing root as well as any bending and twisting, which may have occured along the span.

For incompressible flow, the lift curve slope is only a function of the geometric properties of the wing. In compressible flow, a also becomes a function of Mach number. By varying the Mach number, and thus the lift curve slope, the effects of high speed flow can be evaluated.

Likewise, the torque per unit span can be found from:

$$T(x_2) = q_n ace(\alpha_0 + \theta - h' tan\Lambda) + q_n c^2 c_{mac} \tag{2.57}$$

Definitions of the aerodynamic properties can be found in A.1.

Replacing L and T in Equations (2.48-51) with Equations (2.56-7) gives the complete static aeroelastic system of equations accounting for both the structural and aerodynamic properties of the wing.

2.5 Solving the Static Aeroelastic System

Now that the governing static aeroelastic system of equations is complete, it can be seen from dimensional analysis that h, θ, $\bar{\psi}_2$, and $\tilde{\psi}_2$ are of different dimensions. Thus, for convenience, the system was normalized. A.2 gives the system properties, as well as the quantities used to normalize each.

After the system has been normalized, it can be reduced to two dependent variables: θ and h. θ simply represents the twist about the elastic axis, which has been assumed to correspond to the reference axis. h represents the vertical, or plunging, displacement of the elastic axis. Both are now functions of η, which is the non-dimensionalized spanwise variable, x_2, and runs from 0 to 1. This final system can be written as[5]

$$h^{IV} - m_4 E_1 Q_n h''' + m_3 E_1 Q_n \theta'' - m_{13} Q_n \theta + m_{14} Q_n h' = m_{15} \tag{2.58}$$

$$\theta^{IV} - \frac{4m_{12} - m_6 E_1 Q_n}{m_2 E_1 + 1}\theta'' - m_{16} Q_n \theta + m_{17} Q_n h' - \frac{m_7}{m_2 E_1 + 1} E_1 Q_n h''' = m_{18} + m_{19} \tag{2.59}$$

With the resulting boundary conditions:

$\eta = 0$:

$$m_1 E_1 h''' - m_1 m_4 E_1^2 Q_n h'' + h' + m_1 m_3 E_1^2 \theta' = 0 \tag{2.60}$$

$$m_1 E_1 (m_2 E_1 + 1)\theta''' + [(m_2 E_1 - 1)^2 + m_1 m_6 E_1^2 Q_n]\theta' - m_1 m_7 E_1^2 Q_n h'' = 0 \tag{2.61}$$

$$h = \theta = 0 \tag{2.62}$$

$\eta = 1$:

$$h'' + m_3 E_1 Q_n \theta - m_4 E_1 Q_n h' = -m_5 E_1 \tag{2.63}$$

$$(m_2 E_1 + 1)\theta'' + m_6 E_1 Q_n \theta - m_7 E_1 Q_n h' = -(m_8 + m_9) E_1 \tag{2.64}$$

$$h''' - m_4 E_1 Q_n h'' + m_3 E_1 Q_n \theta' = 0 \tag{2.65}$$

$$(m_2 E_1 + 1)\theta''' - (4m_{12} - m_6 E_1 Q_n)\theta' - m7 E_1 Q_n h'' = 0 \tag{2.66}$$

Here, E_1 is the non-dimensional transverse shear flexibility for a transversely isotropic material, which is defined as the ratio between Young's modulus and the modulus of transverse shear rigidity. If Kirchoff's hypothesis had been assumed, E_1 would equal zero. The m coefficients are functions of the structural and aerodynamic properties of the wing. A.3 gives expressions for the m coefficients. Q_n is the normalized dynamic pressure. When determining divergence speeds, Q_n will become the eigenvalue of the governing equations. When deriving the wing mode shapes, Q_n will take on a value below the critical value of divergence.

Figure 2.2: Simplified geometric description of wing plunge, h, and twist, θ.

Having reduced the system to two dependent variables, h and θ, which vary along the x_2 axis, Fig. 2.1 can be simplified to Fig. 2.2. Notice that both h and θ are descriptions of action in the x_1, X_3 plane, but they are functions of x_2.

Due to the fact that mode shapes become irrelevant quantities after wing failure, the first step in analysis is to determine at what dynamic pressure divergence occurs. The first step in accomplishing this is to create a matrix inclusive of all wing properties. After using *MATHEMATICA*'s DSolve function to solve the governing equations, an 8×8 matrix, Equation (2.67), can be formed from the boundary conditions.

$$[\Delta]\{\beta\} = \{F\} \tag{2.67}$$

Where Δ is the matrix describing the wing structural, geometric, and aerodynamic properties dependent on airflow, β is a vector made up of the 8 unknown boundary conditions, and F are the wing properties independent of airspeed. Knowing that Q_n, the normalized dynamic pressure, is indicative of airspeed and density, it is left as a variable. This now becomes a simple eigenvalue problem, and the divergence dynamic pressure can be obtained by setting the determinant of the matrix equal to zero and solving for Q_n.

In reality the speed of *MATHEMATICA* forces the use of a guess and check technique. By creating a loop, which inputs various values of Q_n before solving the governing equations, the general trend can be used to find where the determinant equals zero.

In order to solve for the bending shapes, the basic technique of matrix inversion was used. In such a large matrix there is the a good chance of this causing an ill-conditioned matrix. To avoid that the process of iterative improvement was used.

$$\{\beta\} = [\Delta]^{-1}\{F\} \tag{2.68}$$

By using Equation (2.68), the unknown boundary conditions could be determined. Plugging these back into the governing equations and choosing a dynamic pressure creates two equations, $H(\eta)$ and $\theta(\eta)$, which describe the pitching and plunging motion of the wing as functions of the spanwise location. To generate truly useful data, these equations can be combined into a single equation describing the effective angle of attack of the wing, α_{eff}, as follows.

$$\alpha_{eff} = \alpha_0\Big(1 + \frac{\theta - \frac{tan(\Lambda)}{AR_1}h'}{\alpha_0}\Big) \tag{2.69}$$

The *MATHEMATICA* code for both the divergence speed and effective angle of attack calculations can be found in A.4 and A.5, respectively.

After determining values for both the divergence speed and effective angle of attack distributions at different Mach numbers, an accurate assessment of the effects of transverse shear deformation in steady flow can be made. The next step in the process is to look at unsteady, oscillatory flow.

2.6 Unsteady Aerodynamics

While steady aerodynamics follows basic principles of physics and can be taught at the undergraduate level, the topic of unsteady aerodynamics requires a good deal more effort. Utilizing higher level math and complex functions, it is an area of study worthy of a dedicated graduate level course. However, the unsteady data necessary in the context of this analysis merely requires calculating values for lift and moment from known equations. Somewhat complicating the process are the Mach number effects.

In order to determine accurate values for the aerodynamic forces, the unsteady equations must be corrected for compressibility. An amazingly complex process to derive, the work can be credited to two men Theodore Theodorsen[1,189] and P.F. Jordan[10,1]. In the early 20th century, Theodorsen derived a complex function, $C(k) = F(k) + iG(k)$, used in the

prediction of unsteady incompressible aerodynamic forces, where k is the reduced frequency of oscillation. This was a giant leap for aerodynamicists and mathematicians at the time. Shortly after the Second World War, Jordan took the analysis of unsteady flow a step further by correcting for the Mach number effects due to compressibility. Jordan's work directly corrects the Theodorsen function through:

$$C_{comp} = \frac{F_{comp}}{F_{incomp}} C_{incomp} \tag{2.70}$$

where F_{incomp} is the real part of the Theodorsen function and

$$F_{comp} = \frac{(2l'_\alpha - l'_z) + \frac{k}{2}(2l''_\alpha - l''_z) - \frac{\pi k}{2}}{C_{l\alpha}[1 + (\frac{k}{2})^2]} \tag{2.71}$$

In Equation (2.71), the l variables are known as the Jordan coefficients and can be found in Reference [10]. Once the Theodorsen function has been corrected for compressibility, it can be used in the equations for unsteady aerodynamics.

Since flutter can be described by a neutrally unstable oscillation of a wing at a given frequency, ω, lift and moment must also be represented as harmonic functions with the same frequency. That is:

$$L(\eta; t) = Re(\hat{L}(\eta) e^{(i\omega t)}) \tag{2.72}$$

$$M(\eta; t) = Re(\hat{M}(\eta) e^{(i\omega t)}) \tag{2.73}$$

where $\hat{L}(\eta)$ and $\hat{M}(\eta)$ are complex amplitudes of the unsteady aerodynamic loads and are related to the complex amplitudes of the oscillatory modes

$$h(\eta; t) = Re(\hat{h}(\eta) e^{(i\omega t)}) \tag{2.74}$$

$$\theta(\eta; t) = Re(\hat{\theta}(\eta) e^{(i\omega t)}) \tag{2.75}$$

by the following relations:

$$\hat{L} = \pi\rho c^3\omega^2(\hat{h}L_{hh} + \hat{\theta}L_{h\theta} + \hat{h}L_{hh'} + \hat{\theta}L_{h\theta'}) \tag{2.76}$$

$$\hat{M} = \pi\rho c^4\omega^2(\hat{h}M_{\theta h} + \hat{\theta}M_{\theta\theta} + \hat{h}M_{\theta h'} + \hat{\theta}M_{\theta\theta'}) \tag{2.77}$$

In Equations (2.76-7), the coefficients of the mode shapes are functions of aerodynamic properties of the wing and the Theodorsen function. It is within these coefficients that the correction for compressibility to the Theodorsen function is made. The equations can be found in A.6.

2.7 Solving the Unsteady Aeroelastic System

The process for solving the unsteady aeroelastic system is much the same as that used for the static state. Both are systems of two governing equations and eight boundary conditions. However, a new problem arises in the unsteady system. Instead of just one eigenvalue describing the static instability, there are two for the dynamic instability. Thus, instead of dynamic pressure, the flutter problem is a function of Ω and k, the normalized circular frequency and reduced frequency, respectively. They can be written as:

$$\Omega = \frac{\omega}{\omega_b} \tag{2.78}$$

$$k = \frac{\omega c}{2V} \tag{2.79}$$

Aside from this point, the unsteady system is similar to that of the steady state. Sparing another lengthy derivation, the unsteady model can be written as:

$$W_1(s)H^{(4)}(\eta) + W_2(s)\Theta^{(3)}(\eta) = 0 \tag{2.80}$$

$$W_3(s)H^{(3)}(\eta) + W_4(s)\Theta^{(4)}(\eta) = 0 \tag{2.81}$$

with the boundary conditions at the root:

$$W_5(s)H^{(3)}(0) + W_6(s)\Theta^{(2)}(0) = 0 \tag{2.82}$$

$$W_7(s)H^{(2)}(0) + W_8(s)\Theta^{(3)}(0) = 0 \tag{2.83}$$

$$h(0) = 0 \tag{2.84}$$

$$\theta(0) = 0 \tag{2.85}$$

and the boundary conditions at the tip:

$$W_9(s)H^{(3)}(1) + W_{10}(s)\Theta^{(2)}(1) = 0 \tag{2.86}$$

$$W_{11}(s)H^{(2)}(1) + W_{12}(s)\Theta^{(3)}(1) = 0 \tag{2.87}$$

$$W_{13}(s)H^{(2)}(1) + W_{14}(s)\Theta^{(1)}(1) = 0 \tag{2.88}$$

$$W_{15}(s)H^{(1)}(1) + W_{16}(s)\Theta^{(2)}(1) = 0 \tag{2.89}$$

Here the terms $W(s)$ is used to describe the aerodynamic, geometric, and structural properties of the wing, which are not shown explicitly. H and Θ are the Laplace transforms of $\hat{h}$ and $\hat{\theta}$, where the superscripts are used to show the order of each system.

Instead of using the DSolve command in *MATHEMATICA*, the system was solved using the Laplace transform technique. This involved taking the Laplace transform of the two governing equations, factoring out the H and Θ and solving two equations for two unknowns. Once H and Θ were determined as functions of s, which represents the independent variable in the Laplace domain, the inverse Laplace transform could be taken, giving $\hat{h}$ and $\hat{\theta}$ as functions of η. Now instead of an eighth order system of differential equations, there is a system of two algebraic equations with eight higher order unknowns. This is where the Laplace method shows its true utility.

The Laplace Transform method takes into consideration a function's initial conditions when working with derivatives. As an example, the Laplace Transform of f' is $sF - f(0)$. This fact allows for the four initial conditions to be taken into account within the two governing equations. Once this is done, the system has four unknowns. By this point in the process the two governing equations have become quite large, and a switch from pen and paper derivation to a computerized solved is needed. Again *MATHEMATICA* fills this role quite nicely.

Using *MATHEMATICA* and wing tip boundary conditions, i.e. $\eta = 1$, a matrix can be created by substituting in the governing equations evaluated at the tip. This leaves a four by four matrix similar to the eight by eight matrix used in the steady state problem. The benefit of using the Laplace technique can be seen when manipulating this matrix, as inverting it has a significantly lower chance of causing an ill-conditioned matrix.

This for by four matrix now has six unknowns: the airflow properties k and Ω, as well as four initial conditions $h'(0)$, $h''(0)$, $\theta'(0)$, and $\theta''(0)$. Depending on what type of data is known, this system can be used to either determined flutter speed and frequency or the wing mode shapes.

In order to determine flutter speed and frequency, the Theodorsen method is used. This method is similar to the eigenvalue problem used to solve for divergence speed, but takes two variables into account, k and Ω. At first it would seem that one matrix with two variables could not be solved. However since the matrix is time dependent, its terms have become complex. This means that for the determinant to approach zero, both the real and imaginary parts of each term must vanish. Theodorsen proposed that this could be done by selecting two values of reduced frequency, both close to the assumed value that would cause

the determinant to equal zero. Then the normalized frequency is varied until two values are determined. One where the real part of the determinant equals zero, and one where the imaginary part equals zero. This process is repeated for the other value of k. The flutter eigenvalues at which the flutter determinant becomes zero can then be determined from the intersection. This can better be explained by Fig. 2.3.

Figure 2.3: Theodorsen's method for determining the flutter eigenvalues k and Ω.

In this example, $k = 0.4$ and $\Omega = 3$ would cause the flutter determinant to vanish. At this point, Equations (2.74-5) can be used to solve for the flutter speed, V, and the flutter frequency, ω.

Having now determined the flutter stability boundary corresponding to the maximum speed and oscillatory frequency the wing can sustain, subcritical mode shapes can be determined. Utilizing the same method as outlined in Section 2.5, the matrix can be inverted. Again the process of iterative improvement reduces the chance of creating an ill-conditioned matrix during the inversion process.

2.8 The Physical Meaning of E/G'

Throughout this derivation the term E/G' has been included as a structural property without much explanation. However, it has never been assigned real-world values. Often when performing research it is easy to focus on the specifics and forget why the research is actually being done. This situation can be avoided if real-life results are always an end goal of the research.

In the case of E/G', the analysis looks at values from 0-100. To better understand how these numbers correlate to some real world composits, Table 5.1 gives different materials along with their E/G' values. Some materials such as fiberglass and graphite are household names. Some are not. However, all of these materials are used throughout industry.

Material	E/G'
Fiberglass	7
Boron	23
Graphite	31

Table 2.1: E/G' Values.

Chapter 3

Results and Discussion

Having laid out the mathematics behind the project, the focus can now be shifted to results. The models developed in Chapter 2 are extremely generic models, which can be used to analyze a large variety of wings. As long as a wing fits into a fairly general category, its specific properties can be input into the model and results for critical eigenvalues and subcritical mode shapes can be determined. For this analysis the wings described in above will be used to analyze the effects of transverse shear rigidity on the static and dynamic aeroelastic instabilities in the presence of compressibility effects.

3.1 Wing Models

Once a generic model was developed, it became necessary to gather information on wings of known geometric, aerodynamic, and structural properties. This is necessary in order to have specific wings to analyze.

Wing	Goland's	400R
Span (ft)	40.0	2.90
Chord (ft)	6.0	0.79
AR	6.67	3.69
Thickness	0.1	0.04

Table 3.1: Goland's Wing and the 400R Wing.

The two wings chosen were Goland's wing[11] and the 400R[12] wing. Specific properties of both can be found in Table 3.1. Obviously, the 400R wing is much smaller and was most likely designed for wind tunnel testing. Looking at the non-dimensional properties of aspect ratio and thickness, it can be seen that the 400R wing is much shorter and thinner than Goland's wing. In terms of this analysis, as a structure become thicker it tends to be more affected by transverse shear deformation.

Unfortunately neither of these wings has a very low transverse shear moduli causing $E/G' \approx 0$. Thus both are considered to be made of classical materials. This problem is overcome by simply varying the value of E/G'. As long as the basic geometric and aerodynamic properties of the wings are held constant, varying only a few properties allows for a parametric analysis to be performed on the aeroelastic instability of the wing.

3.2 Test Equipment

3.2.1 MATHEMATICA[9]

All computer work was done using the Silicon Graphics system in the USNA CADIG center. A UNIX based version of *MATHEMATICA* was run. Input could be made via a TELNET connection from outside of the facility. However, direct input into one of the CADIG servers was more efficient.

Although a powerful computational tool and the choice of many mathematicians for symbolic manipulation, *MATHEMATICA* was shown to have some flaws. The main problem encountered was the black box design of the program. Even though *MATHEMATICA* has literally thousands of functions available for symbolic manipulation and numeric analysis, very few allow the user insight into how the functions actually work. While this is acceptable for basic math functions, processes such as matrix inversion and the numerical solver may or may not be operating as the user desires.

3.3 Steady Flow Analysis

3.3.1 Effect of Mach Number on Divergence Speed

Before any meaningful analysis can be performed on the what shape a wing will take at a give air speed, it is necessary to determine at what speed the wing will fail. This is accomplished by the method laid out in Section 2.5 with the Goland wing properties. For this analysis the wing will be swept forward 20 deg. By varying Mach number and the transverse shear rigidity, a set of data points was created, which showed the effects of both on the divergence speed. Fig. 3.1 shows these results.

Divergence spped is measured as the nondimensional value, Q_n, which is a function of airspeed and air density. Notice the decrease in divergence speed as Mach number increases. This is representative of the higher amount of kinetic energy carried in high speed flow. It is important to remember that Mach number can be used to measure how compressed the air has become. When the freestream air is compressed by the wing at higher Mach numbers, it will transfer a greater amount of energy into the structure. The wing on the other hand can only absorb a constant amount of this in the form of strain energy. Thus, for higher Mach number divergence occurs at a lower value of Q_n.

Figure 3.1: Critical dynamic pressure vs. Mach number for various values of transverse shear rigidity.

One may wonder what significance Fig. 3.1 has knowing that both dynamic pressure and Mach number are related to airspeed. To add atmospheric relevance, the two lines drawn in gray represent the range of values possible in the Earth's atmosphere. The upper line represents sea level conditions on a standard day. The lower line represents 50,000 feet on a standard day. This shows that at sea level higher dynamic pressures can be absorbed by the wing, while at 50,000 feet higher Mach numbers can be attained before divergence occurs. These results represent another benefit gained by high performance aircraft at high altitudes.

3.3.2 Effects of Compressibility on Effective Angle of Attack

Having determined the divergence speeds for the Goland wing at various atmospheric conditions and Mach numbers, analysis of the wing deformation can begin. Again using the Goland wing, the effective angle of attack along the wing semi-span was determined at various airspeeds relating approximately to Mach 0.3, 0.7, and 0.8 at sea level. These results are shown in Fig. 3.2.

Figure 3.2: Spanwise distribution of effective angle of attack across wing semi-span.

The first point to be made concerns the effects of compressibility at low airspeeds. Fig. 3.2a shows that at 200 knots, the limit of incompressible flow theory, the compressible analysis actually shows a smaller effective angle of attack distribution across the wing. Even though the difference is almost negligible, a fraction of a degree at the tip, this adds a factor of safety to the design of low speed aircraft whose designers use incompressible flow theory.

For instance, a general aviation aircraft intended for 200 knots, approximately Mach 0.3 at sea level, was most likely designed using incompressible flow theory. When the compressible analysis is performed in this performance region, it shows that there is actually less deformation than previously calculated. As general aviation aircraft companies, such as Cessna and Diamond, move to more and more composite aircraft components, this information could become very useful.

Another interesting occurrence is the divergence speed. Fig. 3.1 showed that for the Goland wing divergence should occur between Mach 0.7 and 0.8. Fig. 3.2 shows this happening. While the jump between an airspeed of 200, Fig. 3.2a, and 460 knots, Fig. 3.2b, only caused an 8 degree change in the effective angle of attack at the wing tip, the jump between 460 and 530 knots, Fig. 3.2c, caused a larger jump in both the compressible and incompressible analysis. This reinforces the fact that for the Goland wing, divergence speed does occur in this range.

Forward swept wings tend to have a lower divergence speed than straight or swept back wings. Forward swept wings also show more deformation than straight or swept back wings at the same airspeed. This fact was a major problem when designing the X-29 Forward Swept Wing aircraft. To alleviate this problem, composite tailoring was used. In contrast, swept back wings experience divergence at much higher speeds than swept forward wings, therefore lessening wing deformation. Fig. 3.3 shows the effective angle of attack across the wing semi-span for Goland's wing swept back 20 deg. Results are shown for airspeeds of 460 and 530 knots, relating to Mach 0.7 and 0.8 at sea level.

Figure 3.3: Effective angle of attack across semi-span for Goland's wing swept back 20deg.

Using Fig. 3.3 it is easier to point out how little changes in airspeed affect wing deformation. While there is a 20 deg to 30 deg jump between 460 and 530 knots for the forward swept wing, there is only a 0.5 deg jump for the swept back wing.

3.3.3 Effect of Sweep on Effective Angle of Attack

Although it is quite simple to state that forward swept wings are deformed more than swept back wings at similar air speeds, further investigation into the effects of sweep produce interesting results. Before these results can be discussed it is necessary to explain the level of influence sweep has on the aerodynamic and geometric properties of a wing.

When air flows across a wing, only the normal component of the airspeed to the leading edge of the wing creates aerodynamic forces and moments. Thus, if an aircraft with wings swept at an angle Λ, is flying at an airspeed V, then the effective airspeed is actually only $V\cos(\Lambda)$. This relationship can be directly applied to Mach number. In truth if an aircraft is flying at Mach 1 with its wings swept back 20 deg, its wing cross-section is only seeing Mach 0.94. Sweep angle also has an effect on the three dimensional lift-curve slope of a wing, a_0. As the magnitude of sweep increases, a_0 also increases by $1/\cos(\Lambda)$. This effect acts opposite to the loss of lift caused by sweep's reduction of effective airspeed.

Geometrically, sweep alters the effective angle of attack as shown in Equation (2.56). α_{eff} is a coupling of the bending and torsional deformations, otherwise known as plunging and pitching modes. Sweep angle is necessary to relate these two. However, instead of using the cosine of sweep, this coupling relies on the tangent of sweep. Knowing that the tangent function diverges at 90deg and -90deg, special attention must be paid to its effect on the results. Obviously as sweep approaches these values, the two-dimensional aerodynamic theory used in the model will no longer work.

With all these sweep effects working with and against each other, as both cosine and tangent functions, it is hard to predict exactly what overal effect sweep will have. Fortunately, computers can perform these complicated calculations in seconds and output the results. From that Fig. 3.4 shows the effects of sweep on effective angle of attack. Positive values of Λ indicate a swept back wing and negative values indicate a forward swept wing.

Figure 3.4: Effect of sweep on effective angle of attack at 550 knots.

At first glance, the results for the swept back appear reversed. Not only is the magnitude smaller, but the wing actually has a lower angle of attack at the tip than the root. This is due to $\tan(\Lambda)$ being positive for a swept back wing. Even though the twist pulls the wing tip up, the coupled effect of the wing plunging mode causes an overall decrease in the effective angle of attack. For a swept back wing these two offset one another, but in swept forward wings they both work together. This likely explains the fact that forward swept wings diverge at lower air speeds.

The truly interesting results come from the different values of forward sweep. Starting from 10 deg, increasing forward sweep causes an increase in effective angle of attack. However, this trend reverses somewhere between $\Lambda = 30deg$ and $\Lambda = 50deg$. By the time $\Lambda = 60deg$ the deformation is back below $\Lambda = 20deg$. This occurrence is most likely due to the switch between a cosine dominant function and a tangent dominated function. At -30 deg cosine is 0.87 and tangent is -0.58, by -50 deg cosine is 0.64 while tangent is 1.19.

This switch between dominant terms in Equation (2.56) signifies two points. The first is that at a certain sweep angle, geometric properties become more important than purely aerodynamic principles. The second, which is less encouraging, is that at a certain point, the equations may become unrealistic. To validate results at this transition point experimental data would have to be collected. If indeed this data did not show this region of transition, then a line would have to be drawn showing where the theory no longer held true.

3.3.4 Effects of Transverse Shear Deformation on Effective Angle of Attack

Having analyzed the effects of compressibility and sweep on the steady state aeroelastic properties on Goland's wing, an accurate discussion of the effects of transverse shear can now be made. It was already shown in Section 5.1.1 that as E/G' increased, divergence speed decreased. This made sense as more flexibility through the wing's thickness would accelerate wing instability. Now comparing the effect of transverse shear deformation on effective angle of attack, more in depth conclusions can be made. Fig. 3.5 shows the spanwise effective angle of attack distribution for various values of E/G' for both incompressible and compressible flows.

Figure 3.5: Effect of transverse shear on effective angle of attack at 200 knots.

Again similar to the results drawn from the analysis of divergence speeds, it can be seen from the analysis of Fig. 3.5 that the more flexible the wing in transverse shear, the more it will deform at a certain air speed. This can directly be related to a lower divergence speed for flexible wings.

Also from Fig. 3.5a it can be seen that at 200 knots, incompressible analysis is fairly similar for all values of E/G'. There is less than a 1 % difference in the effective angle of attack at the wing tip. However, the compressible analysis shows that a much larger gap in bending occurs between the different values of E/G'. An exact physical explanation for this is difficult to draw. One explanation may be that as the wing bends, compressible flow has a high tendency to increase the aerodynamic loads and moments it places on the wing. Thus, a change in wing flexibility would have a greater effect in compressible flow, than in incompressible flow. It is important to remember that these results and hypothesis are for low speed flows. Because as Fig. 3.6 points out, varying three properties makes an analysis of results very difficult.

Figure 3.6: Effect of transverse shear on effective angle of attack at 460 knots.

Fig. 3.6 shows that even at high airspeeds incompressible theory causes a larger deformation than compressible theory. However, the gap is definitely closing. At values of $E/G' = 0$ there is hardly any difference between the compressible and incompressible analysis. It is quite possible that at higher airspeeds and thus, higher Mach numbers, the magnitude of the compressible aerodynamic loads begins to catch up with that of the incompressible loads.

To further back up this argument, the large deformation gap that was seen in the compressible analysis at 200 knots has been significantly reduced. Fig. 3.6 actually shows that the incompressible theory now has a larger gap between the wing tip deformations of $E/G' = 0$ and $E/G' = 100$. To explain this phenomenon better, deeper research into compressible subsonic aerodynamics would be necessary. As these results seem to go against intuition, it appears that this is simply a case of finding answers, which lead to more questions.

3.4 Flutter Analysis

3.4.1 The Influence of Transverse Shear Deformation on Flutter Frequency

Transitioning now to the unsteady side of the field, the focus switches from divergence speeds and effective angles of attack to flutter frequencies and velocities. It is important to remember throughout this analysis that wing failures can occur through both static aeroelastic instability, divergence, and dynamic instability, flutter. In different configurations and environments, either type of failure could occur first. Thus, determining the conditions necessary for both categories to occur is vital in understanding what flow regimes an aircraft can operate in.

Moving on, Theodorsen's method was utilized to investigate the effects of transverse shear rigidity and Mach number on the flutter eigenvalues. Unlike the static instability described by a single eigenvalue, the flutter instability is described by two eigenvalues. Both of these eigenvalues were normalized during this analysis in order to more eliminate the necessity of carrying units. As flutter frequency, rate of wing oscillation at failure, is seemingly more difficult to understand it will be dealt with first. Fig. 3.7 shows the effects of transverse shear rigidity.

Figure 3.7: Effect of transverse shear flexibility on flutter frequency.

From this it can be seen that as the wing becomes more flexible, its flutter frequency decreases at all Mach numbers. However, as the Mach number increases up into the high speed subsonic and supersonic regions the curves become more linear. This follows the hypothesis from steady flow that transverse shear rigidity has a larger impact upon wings in low speed subsonic flow. It is also clear that as the wing becomes more flexible in transverse shear flutter occurs at lower frequencies. This phenomenon is consistant with the inability of the wing structure to dissipate the energy absobed due to the aerodynamic loads.

Figure 3.8: Effect of Mach number on flutter frequency.

To better illustrate this idea, Fig. 3.8 displays flutter frequency against Mach number for three values of E/G'. If Fig. 3.7 left any doubt that transverse shear deformation is a problem at low airspeeds, Fig. 3.8 will erase it.

The results show the importance of Mach number on flutter frequency. Notice the large difference between flutter frequencies at Mach 0. The wing with $E/G' = 10$ has a flutter frequency approximately 20% larger than the $E/G' = 100$ wing. As the Mach number increases, this difference decreases. At Mach 1 this difference has fallen to only 12%, and at Mach 2 it is well under 10%. Again, the effects of transverse shear deformation on the flutter frequency are greater at the lower Mach numbers.

Fig. 3.8 also draws an excellent picture of what is commonly known as "transonic dip". Notice the sudden decrease in the flutter frequency as the Mach number crosses from 0.95 to 1.2. Past Mach 1.2 the curve again levels out. This occurrence is common in transonic aerodynamic and gives shows theoretically just how dynamic the transonic regime can be.

3.4.2 The Influence of Transverse Shear Deformation on Flutter Speed

Flutter speed, the velocity at which dynamic instability occurs, can be dealt with much like flutter frequency. Exhibiting many of the same tendencies when transverse shear rigidity and Mach number are altered, the flutter speed is much simpler to physically comprehend. Basically above a certain airspeed, an oscillating wing will no longer be able to dissipate the aerodynamic energy absorbed by the wing structure. When the structure absorbs energy at a rate higher than it can dissipate that energy, wing oscillations will increase in amplitude until structural failure occurs. The flutter speed is the neutrally stable speed, which separates damped oscillations from undamped oscillations.

In the test flights of many aircraft certain "danger zones" are encountered when the flutter speed is reached. However, often higher air speeds can be attained simply by changing altitude and circumventing the danger zone. If a change in altitude can be used to avoid certain flutter speeds it is not unlikely that Mach effects have something to do with flutter speeds. Fig. 3.9 shows the effects of tranverse shear flexibility on flutter speed for various Mach numbers.

Figure 3.9: Effect of transverse shear on flutter speed.

Similar to its effect on flutter frequency, an increase in transverse shear tends to lower the flutter speed. It is interesting to note, that as E/G' becomes large, the Mach lines seem to converge upon one another. All subsonic lines seem to curve up towards an imaginary tangential line, while the supersonic line curves down towards the same line. It turns out that this line seems to be exactly where the Mach 1 results would be depicted. Unfortunately, aerodynamic data for Mach 1 is extremely unstable and thus this point cannot be definitively proven. However, drawing a straight line from where Mach 1 should cross the dependent axis to the convergence of the other four curves, it does not seem unreasonable that the curves could be converging to Mach 1.

In gas dynamics both subsonic and supersonic flows tend to approach Mach 1 at the throat of nozzles. Could it be that flutter speed exhibits the same properties? It looks as if both the subsonic and supersonic curves are converging towards the imaginary Mach 1 line as E/G' increases. Further investigation of this behavior could produce very interesting results.

Getting back to the effects of Mach number on flutter speed, Fig. 3.10 shows once again the inverse effect of Mach number on aeroelastic properties. As the Mach number increases, the difference between flutter speeds for various values of E/G' decreases. Similar to Fig. 3.1, the grey atmospheric boundary lines have been placed on the plot. The left line represents sea level conditions, while the right line represents 50,000 ft. Notice that for flutter speed, the atmospheric range is much smaller than for divergence speed.

Figure 3.10: Effect of Mach number on flutter speed.

As mentioned before, the three curves converge as Mach number is increased up to a point. Fig. 3.10 actually shows a slight divergence between the curves past Mach 1.2. This could be due to the $E/G' = 10$ curve showing somewhat erratic behavior, or possibly a new phenomenon not yet seen.

The inclusion of the atmospheric boundary lines shows where possible inaccuracies could have occurred in a merely incompressible analysis. By the time the flutter speed curves reach the first boundary line, sea level, they have already decreased almost 10%. Similar to the trend shown in the divergence speed analysis, the flutter speed decreases as Mach number increases. This could be due to the increased energy in the compressed flow.

3.4.3 Comparison Between Goland's Wing and the 400R Wing

Having looked at what effects both Mach number and transverse shear flexibility have on the aeroelastic properties of Goland's wing, the next step is to look into the effects that wing geometry has on the flutter eigenvalues. As shown in Table 3.1, the 400R wing is a thinner wing than Goland's wing. It also has a lower aspect ratio. Fig. 3.11 shows the comparison between the flutter frequencies for both Goland's wing and the 400R wing.

Notice that as the tranverse shear flexibility is increased, the flutter frequency of Goland's wing is more affected. Stepping back a moment to think about this, the answer to why this occurs becomes apparent. Transverse shear is a measure of the deformation occurring through the thickness of the wing. If one wing is thicker than another, such as is the case with Goland's wing and the 400R, it would make sense that the thicker wing is more affected by a change in transverse shear flexibility.

Figure 3.11: Comparison of flutter frequencies for Goland's wing and the 400R.

The effects of wing thickness on flutter speed produce much the same results. Fig. 3.12 shows the flutter speed comparison.

Again an increase in the value for E/G' produces a noticeable change in the flutter speed for Goland's wing. However, almost no change occurs to the characteristics of the 400R. Again this should concern general aviation enthusiasts more than high performance designers. Most low speed aircraft utilize a thick wing to generate lift at low airspeeds. High performance jets can create most of their lift through airspeed and require small thickness and camber in their wings. Again it turns out that transverse shear deformation affects general aviation aircraft more than their high performance counterparts.

Figure 3.12: Comparison of flutter speeds for Goland's wing and the 400R.

3.5 Summary and Conclusions

Initially, this research was focused on analyzing the aeroelastic effects of compressibility and transverse shear deformation with the thought that the findings could be used to design high-performance aircraft. However, as the results have shown it is quite obvious that transverse shear is a larger problem at low air speeds. Every analysis from the effect of transverse shear flexibility to wing thickness showed that the wing properties common in general aviation aircraft led to larger variations in aerodynamic performance.

This is not to say that high performance aircraft should have no concern of aeroelastic failure. It was shown that as Mach number increases all critical speeds decrease, which works against the jet community. Nevertheless, as all aviation communities from "fighter jocks" to civilian student pilots continue the search for lighter, faster, and more dependable aircraft, composite structures will no doubt be used in place of their metallic ancestors.

It is important to remember that although theoretical analysis has its flaws, it is useful as a low-cost method of determining trends in data. Specific trends discerned from this analysis are:

a. At low air speeds, and thus low Mach numbers, variations in transverse shear flexibility had more of an effect on a wing's aeroelastic instabilities than at high speeds.

b. As Mach number increases, the critical air speeds in both steady and unsteady flow decrease.

c. Compressible analysis varies from incompressible analysis by anywhere from 0-30% in standard atmospheric conditions.

3.6 References

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Appendix A

Table A.1: Definitions of aerodynamic characteristics.

$q_n = \frac{1}{2}\rho_\infty V_\infty^2 cos^2\Lambda$
$a =$ lift curve slope $a_0 =$ incompressible lift curve slope $a_0 = \frac{2\pi}{1+\frac{4}{AR}cod\Lambda}$ $\frac{dC_l}{d\alpha} =$ compressible lift curve slope $\frac{dC_l}{d\alpha} = \frac{a_0 cos\Lambda}{\sqrt{1-M_0^2 cos^2\Lambda + \left(\frac{a_0 cos\Lambda}{\pi AR}\right)^2} + \frac{a_0 cos\Lambda}{\pi AR}}$
$AR = \frac{b^2}{S}$
$l = \frac{b}{2}$

Table A.2: Non-dimensionalized values.

$$\eta = \frac{x_2}{l}$$

$$\frac{d}{dx_2} = \frac{1}{l}\frac{d}{d\eta}$$

$$\frac{d^2}{dx_2^2} = \frac{1}{l^2}\frac{d^2}{d\eta^2}$$

$$h = \frac{h}{c}$$

$$\bar{e} = \frac{e}{c}$$

$$\bar{t} = \frac{t}{c}$$

$$\bar{\theta} = \theta$$

$$\bar{x}_0 = \frac{x_0}{c}$$

$$E_1 = \frac{E}{G'}$$

$$AR_1 = \frac{AR}{2}$$

$$Q_n = 12q_n\frac{AR_1^3(1-\nu^2)}{E\bar{t}^3}$$

Table A.3: Determination of 'm' coefficients.

$m_1 = \frac{\bar{t}^2}{12(1-\nu^2)AR_1^2}$	
$m_2 = \frac{\bar{t}^2}{2(1+\nu)}$	$m_{12} = \frac{m_2}{m_1}$
$m_3 = m_1 a_0 AR_1$	$m_{13} = \frac{m_3}{m_1}$
$m_4 = m_1 a_0 tan\Lambda$	$m_{14} = \frac{m_4}{m_1}$
$m_5 = m_1 a_0 \alpha_0 AR_1 Q_n$	$m_{15} = \frac{m_5}{m_1}$
$m_6 = \frac{\bar{t}^2}{(1-\nu^2)AR_1}\bar{e}a_0$	$m_{16} = \frac{m_6}{m_1}$
$m_7 = m_6 \frac{tan\Lambda}{AR_1}$	$m_{17} = \frac{m_7}{m_1}$
$m_8 = m_6 \alpha_0 Q_n$	$m_{18} = \frac{m_8}{m_1}$
$m_9 = \frac{\bar{t}^2}{(1-\nu^2)AR_1} c_{mac} Q_n$	$m_{19} = \frac{m_9}{m_1}$

Table A.4: *MATHEMATICA* code for Divergence Speed.

```
Qnlist = {};                                          % Define nullset matrix

Do[                                % Begin loop to cycle through values of Q_n
pois = .25;                                           % Set Poison's ratio
E1 = 70;                                              % Set E/G'

Clear[h];
Clear[theta];
Clear[equations];                                     % Clear all variables
Clear[A];
Clear[sol];

sweep = -.3509;                                       % Sweep
a0 = 6.28/(1 + 4/8 Cos[sweep]);                       % Incompressible lift curve slope
AR = 8;                                               % Aspect ratio
AR1 = AR/2;                                           % Semi-span aspect ratio

alpha0 = 0.0873;                                      % Angle of attack
alphabar = alpha0;                                    % Angle of attack (1st normalization)
alphadbar = alpha0;                                   % Angle of attack (2nd normalization)

ebar = .1;                                            % Elastic offset
tbar = .1;                                            % Wing thickness

cmac = 0;                                             % Moment coefficient

m1 = (tbar)^2/(12 (1 - (pois)^2) (AR1)^2);
m2 = (tbar)^2/(2 (1 + pois));
m3 = m1 a0 AR1;
m4 = m1 a0 Tan[sweep];
m5 = m1 a0 alphabar AR1 Qn;                % Define coefficients from governing equations
m6 = ((tbar)^2/(AR1 (1 - (pois)^2)))*ebar a0;
m7 = m6 Tan[sweep]/AR1;
m8 = m6 alphadbar Qn;
m9 = ((tbar)^2/((1 - (pois)^2) AR1))*cmac Qn;
```

```
m12 = m2/m1;
m13 = m3/m1;
m14 = m4/m1;
m15 = m5/m1;                              % Define coefficients from governing equations
m16 = m6/m1;
m17 = m7/m1;
m18 = m8/m1;
m19 = m9/m1;

a1 = -m4 E1 Qn;
a2 = m3 E1 Qn;
a3 = -m13 Qn;                             % Define coefficients from governing equations
a4 = m14 Qn;
b1 = m15;

a5 = -(4 m12 - m6 E1 Qn)/(m2 E1 + 1);
a6 = -m16 Qn;
a7 = m17 Qn;                              % Define coefficients from governing equations
a8 = -(m7/(m2 E1 + 1)) E1 Qn;
b2 = m18 + m19;

c1 = m1 E1;
c2 = -m1 m4 E1^2 Qn;
c3 = 1;
c4 = m1 m3 E1^2 Qn;                       % Define coefficients from governing equations
c5 = m1 E1 (m2 E1 + 1);
c6 = ((m2 E1 + 1)^2 + m1 m6 E1^2 Qn);
c7 = -m1 m7 E1^2 Qn;

d1 = m3 E1 Qn;
d2 = -m4 E1 Qn;
b3 = -m5 E1;
d3 = (m2 E1 + 1);;
d4 = m6 E1 Qn;
d5 = -m7 E1 Qn;                           % Define coefficients from governing equations
b4 = -(m8 + m9) E1;
d6 = -m4 E1 Qn;
d7 = m3 E1 Qn;
d8 = (m2 E1 + 1);
d9 = -(4 m12 - m6 E1 Qn);
d10 = -m7 E1 Qn;
```

```
sol = DSolve[
{ h''''[y] + a1 h'''[y] + a2 theta''[y]
+ a3 theta[y] + a4 h'[y] == b1,
theta''''[y] + a5 theta''[y]                 % Solve governing equations
+ a6 theta[y] + a7 h'[y] + a8 h'''[y] ==
b2}, {h[y], theta[y]}, y];

h[y_] = First[h[y] /. sol];
theta[y_] = First[theta[y] /. sol];
equations = {h[0] == 0,
theta[0] == 0,
c1 h'''[0] + c2 h''[0]
+ c3 h'[0] + c4 theta'[0] == 0,
c5 theta'''[0] + c6 theta'[0]
+ c7 h''[0] == 0,                            % Input boundary conditions
h''[1] + d1 theta[1] + d2 h'[1] == b3,
d3 theta''[1] + d4 theta[1] ,
+ d5 h'[1] == b4
h'''[1] + d6 h''[1] + d7 theta'[1] == 0,
d8 theta'''[1] + d9 theta'[1]
+ d10 h''[1] == 0} ;

equations = Simplify[equations];                        % Simplify system
A = Table[Coefficient[equations[[i, 1]],     % Create matrix of coefficients
C[j]], {i, 1, 8}, {j, 1, 8}];

detA = Det[A];                                    % Calculate determinant
Print[detA];                                         % Output determinant
Qnlist = Append[Qnlist, detA],               % Create list of determinants
{Qn, X, Y, Z}]                               % Run loop from X to Y by Z
```

Table A.5: *MATHEMATICA* code for Effective AOA.

```
Clear[h];
Clear[theta];
Clear[equations];
Clear[coeffs];
Clear[sol];                                                    % Clear all variables
Clear[Theta];
Clear[H];
Clear[aeff];

Youngs = Input["Young's Modulus = "];                          % Input Young's modulus
G = Input["Modulus of Rigidity = "];                           % Input shear modulus
pois = Input["Poisson's Ratio = "];                            % Input Poison's ratio
E1 = Youngs/G;                                                 % Calculate E/G'
comp = Input["1) Compressible                                  % Choose flow regime
Analysis, 2) Incompressible Analysis"];

alphain = Input["Angle of                                      % Input AOA in deg
Attack(deg) = "];
alpha0 = alphain * 3.14159/180;                                % Degree to radians
alphabar = alpha0;                                             % Angle of attack (1st normalization)
alphadbar = alpha0;                                            % Angle of attack (2nd normalization)

c = Input["Chord (ft) = "];                                    % Input chord
b = Input["Wing Span (ft) = "];                                % Input wingspan
l = b/2;                                                       % Calculate semispan
e = Input["Elastic Offset(ft) = "];                            % Input elastic offset
ebar = e/c;                                                    % Calculate elasic offset
t = Input["Wing Thickness(ft) = "];                            % Input wing thickness
tbar = t/c;                                                    % Calculate wing thickness

Sweep = Input["Wing                                            % Input sweep
Sweepback(deg) = "];
sweep = Sweep * 3.14159/180;                                   % Degree to radians
AR = b/c;                                                      % Calculate aspect ratio
AR1 = AR/2;                                                    % Calculate semi-aspect ratio
V = Input["Airspeed(ft/s) = "];                                % Input velocity
roe = Input["Density(slugs/ft^3) = "];                         % Input density
T = Input["Temperature(deg F) = "];                            % Input temp
sound = (1.4*1716*(T + 460))^.5;                               % Calculate speed of sound
M = V/sound;                                                   % Calculate Mach number
```

```
qn = .5 * roe * V^2 * Cos[sweep]^2;                % Calculate dynamic pressure
Qn = qn * 12 * (1 - pois^2) *                      % Normalize dynamic pressure
AR1/(Youngs * tbar^3);

aincomp = 6.28/(1 + (4/AR) Cos[sweep]);            % Incompressible lift curve slope
acomp = (aincomp*Cos[sweep])/((1 -
M^2*Cos[sweep]^2 + (aincomp*Cos[sweep]/            % Compressible lift curve slope
(3.14159*AR))^2)^.5 +
aincomp*Cos[sweep]/(3.14159*AR));

cmac = 0;                                          % Moment coefficient

If[comp == 2, a0 = aincomp, a0 = acomp];           % Choose flow regime

m1 = (tbar)^2/(12 (1 - (pois)^2) (AR1)^2);
m2 = (tbar)^2/(2 (1 + pois));
m3 = m1 a0 AR1;
m4 = m1 a0 Tan[sweep];
m5 = m1 a0 alphabar AR1 Qn;                        % Define coefficients from governing equations
m6 = ((tbar)^2/(AR1 (1 - (pois)^2)))*ebar a0;
m7 = m6 Tan[sweep]/AR1;
m8 = m6 alphadbar Qn;
m9 = ((tbar)^2/((1 - (pois)^2) AR1))*cmac Qn;

m12 = m2/m1;
m13 = m3/m1;
m14 = m4/m1;
m15 = m5/m1;                                       % Define coefficients from governing equations
m16 = m6/m1;
m17 = m7/m1;
m18 = m8/m1;
m19 = m9/m1;

a1 = -m4 E1 Qn;
a2 = m3 E1 Qn;
a3 = -m13 Qn;                                      % Define coefficients from governing equations
a4 = m14 Qn;
b1 = m15;
```

```
a5 = -(4 m12 - m6 E1 Qn)/(m2 E1 + 1);
a6 = -m16 Qn;
a7 = m17 Qn;                          % Define coefficients from governing equations
a8 = -(m7/(m2 E1 + 1)) E1 Qn;
b2 = m18 + m19;

c1 = m1 E1;
c2 = -m1 m4 E1^2 Qn;
c3 = 1;
c4 = m1 m3 E1^2 Qn;                   % Define coefficients from governing equations
c5 = m1 E1 (m2 E1 + 1);
c6 = ((m2 E1 + 1)^2 + m1 m6 E1^2 Qn);
c7 = -m1 m7 E1^2 Qn;

d1 = m3 E1 Qn;
d2 = -m4 E1 Qn;
b3 = -m5 E1;
d3 = (m2 E1 + 1);;
d4 = m6 E1 Qn;
d5 = -m7 E1 Qn;                       % Solve governing equations
b4 = -(m8 + m9) E1;
d6 = -m4 E1 Qn;
d7 = m3 E1 Qn;
d8 = (m2 E1 + 1);
d9 = -(4 m12 - m6 E1 Qn);
d10 = -m7 E1 Qn;

sol = DSolve[
{ h''''[y] + a1 h'''[y] + a2 theta''[y]
+ a3 theta[y] + a4 h'[y] == b1,
theta''''[y] + a5 theta''[y]          % Solve governing equations
+ a6 theta[y] + a7 h'[y] + a8 h'''[y] ==
b2}, {h[y], theta[y]}, y];
```

```
h[y_] = First[h[y] /. sol];
theta[y_] = First[theta[y] /. sol];
equations = {h[0] == 0,
theta[0] == 0,
c1 h"'[0] + c2 h"[0]
+ c3 h'[0] + c4 theta'[0] == 0,
c5 theta"'[0] + c6 theta'[0]
+ c7 h"[0] == 0,                              % Input boundary conditions
h"[1] + d1 theta[1] + d2 h'[1] == b3,
d3 theta"[1] + d4 theta[1] ,
+ d5 h'[1] == b4
h"'[1] + d6 h"[1] + d7 theta'[1] == 0,
d8 theta"'[1] + d9 theta'[1]
+ d10 h"[1] == 0} ;

coeffs = Solve[equations, {C[1], C[2], C[3],  % Solve for unknowns
C[4], C[5], C[6], C[7], C[8]}];
Theta[y_] = Chop[ComplexExpand[Simplify       % Solve for twist
[First[theta[y] /. coeffs]]]];
H[y_] = Chop[ComplexExpand[Simplify           % Solve for bending
[First[h[y] /. coeffs]]]];
aeff[y_] = 1 + (Theta[y] -
(Tan[sweep]/AR1) D[H[y], {y, 1}])/alpha0;     % Solve for effective AOA

plotA = Table[{l*y, Re[H[y]*c]},              % Eliminate imaginary term
{y, 0, 1, .01}];
plotB = Table[{l*y, Re[Theta[y]]},            % Eliminate imaginary term
{y, 0, 1, .01}];
plotC = Table[{l*y, Re[aeff[y]]*alphain},     % Eliminate imaginary term
{y, 0, 1, .01}];

ListPlot[plotA, PlotJoined -¿ True,           % Plot bending
AxesLabel -¿ {"Non-Dimensional Semispan",
"Vertical Displacement (ft)"}];
ListPlot[plotB, PlotJoined -¿ True,           % Plot twist
AxesLabel -¿ {"Non-Dimensional Semispan",
"Twist (deg)"}];
ListPlot[plotC, PlotJoined -¿ True,           % Plot effective AOA
AxesLabel -¿ {"Non-Dimensional Semispan",
"Effective Angle of Attack (deg)"}];
```

Table A.6: Unsteady aerodynamic coefficients.

$$L_{hh} = 2\left(1 + 2\tfrac{G}{k}\right) - 4\tfrac{F}{k}i$$

$$L_{h\theta} = 2a + 2\tfrac{F}{k^2} - 4\tfrac{G}{k}\left(\tfrac{1}{4} - a\right) + \tfrac{1}{k}\left[\left(1 + 2\tfrac{G}{k} + 4\left(\tfrac{1}{4} - a\right)\right)\right] i$$

$$L_{hh'} = -4\tfrac{F}{k^2 AR}\tan\Lambda - \tfrac{2}{kAR}\tan\Lambda\left(1 + 2\tfrac{G}{k}\right) i$$

$$L_{h\theta'} = 4\tfrac{F}{k^2 AR}\left(\tfrac{1}{4} - a\right)\tan\Lambda - \tfrac{2}{kAR}\tan\Lambda\left[a - \tfrac{2}{k}\left(\tfrac{1}{4} - a\right)G\right] i$$

$$M_{\theta h} = 4a + 8\tfrac{G}{k}\left(\tfrac{1}{4} + a\right) - 8\tfrac{F}{k}\left(\tfrac{1}{4} + a\right) i$$

$$M_{\theta\theta} = \tfrac{1}{8} + 4a^2 + 4\tfrac{F}{k^2}\left(\tfrac{1}{4} + a\right) - 8\tfrac{G}{k}\left(\tfrac{1}{16} - a^2\right) - \tfrac{2}{k}\left[\tfrac{1}{4} - a - 2\tfrac{G}{k}\left(\tfrac{1}{4} + a\right) - 4F\left(\tfrac{1}{16} - a^2\right)\right] i$$

$$M_{\theta h'} = 8\tfrac{F}{k^2 AR}\tan\Lambda\left(\tfrac{1}{4} + a\right) - \tfrac{4}{kAR}\tan\Lambda\left[a + 2\tfrac{G}{k}\left(\tfrac{1}{4} + a\right)\right] i$$

$$M_{\theta\theta'} = \tfrac{-1}{k^2 AR}\tan\Lambda\left[1 - 16F\left(\tfrac{1}{16} - a^2\right)\right] - \tfrac{4}{kAR}\tan\Lambda\left[\tfrac{1}{32} + a^2 - 2\tfrac{G}{k}\left(\tfrac{1}{16} - a^2\right)\right] i$$